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Optimal performance of an endoreversible cycle operating between a heat source and sink of finite capacities

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Abstract. Optimal performance of an endoreversible cycle operating between a finite heat source and sink is studied systematically. First, the fundamental optimal relation of the cycle is derived. Then, the relation is used to deduce the performance bounds and some significant discussion is presented. The results obtained here are more general and useful than those of two infinite heat source cycles. Thus, they may contribute to a better understanding of the optimal performance of real heat engines.

1. Introduction

Since finite-time thermodynamics was advanced, many authors have studied the optimal performance of endoreversible two infinite heat source cycles. The fundamental optimal relation and many important performance bounds of the cycle are derived [1–12]. A relatively systematic theory of the two heat source endoreversible cycle, which has more guiding significance than classical thermodynamic theory for a real engine or refrigerator, is set up. Some authors have also studied the endoreversible finite heat source cycle [13–15]. The fundamental optimal relation and some performance bounds of a cycle with a finite heat source and an infinite heat sink are derived [14, 15]. The expression of the maximum work per cycle for a two finite heat source cycle is derived and the cycle producing the maximum work is illustrated [13]. So far, however, the study of the two finite heat source cycle is not as good as that of the two infinite heat source cycle. For example, the study of the fundamental optimal relations and performance bounds of a class of cycles, such as the Diesel cycle, Brayton cycle, and so on, is still relatively rare. In fact, finite heat reservoirs are commonplace features of real energy conversion systems, we should attach importance to it in the researches of thermodynamic cycle theory. The fundamental optimal relation is the kernel of finite-time thermodynamic cycle theory [16].

In this paper, we study the optimal performance of a cycle operating between a finite heat source and sink. It is assumed that there exists only the irreversibility of heat conduction between the working fluid and heat sources (i.e. the cycle is endoreversible) and that heat transfer obeys Newton's law. The fundamental optimal relation of the cycle is derived, and the influences of finite heat capacities of the sources are expounded. Some new performance bounds and significant conclusions which are more general and useful than those of endoreversible two infinite or only one finite heat source cycles are obtained. All

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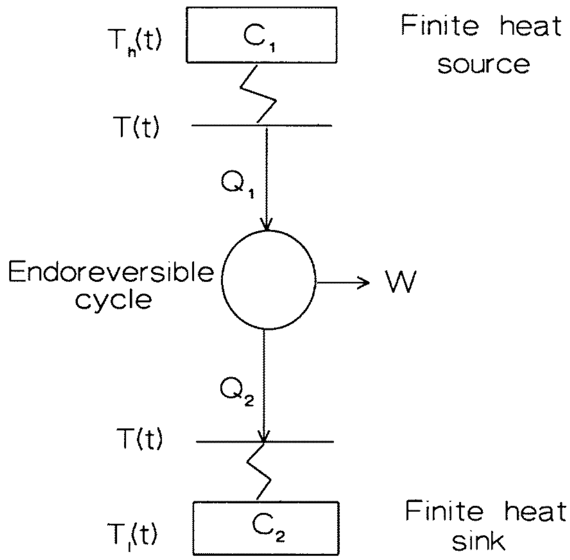

Figure 1. Schematic diagram of an endoreversible two finite heat source cycle.

the conclusions concerning the endoreversible cycles with two infinite heat sources and with a finite and an infinite heat source can be deduced from them. Thus they form a new theory of the endoreversible two heat source cycles and may serve as a guide to the evaluation of existing engines and the optimal design of future engines.

2. Cycle model

The model of the cycle adopted here is shown schematiclly in figure 1. The cycle operates in a fixed cyclic time τ. The working fluid in the cycle is alternately connected to a hot source with constant heat capacity C_1 and initial temperature T_H and to a cold reservoir with constant heat capacity C_2 and initial temperature T_L for each cycle. The two steps in the cycle during which the working fluid is disconnected from one reservoir and connected to the other are taken to be reversibly adiabatic. It is assumed that these steps occur instantaneously in which the temperature of the working fluid $T(t)$ changes discontinuously [13, 17]. The only irreversibility in the cycle results from the finite rate of heat conduction between the working fluid and reservoirs, while a reversible cycle inside the working fluid is carried out.

According to Newton's law, the heat Q_1 absorbed from the hot source and the heat Q_2 released to the cold reservoir by the working fluid per cycle are given by

$$Q_1 = \int_0^\tau K_1(t)[T_h(t) - T(t)]\,\mathrm{d}t \tag{1}$$

$$Q_2 = \int_0^\tau K_2(t)[T(t) - T_l(t)]\,\mathrm{d}t \tag{2}$$

where $T_h(t)$ and $T_l(t)$ are, respectively, the temperatures of the hot source and the cold reservoir, and $K_1(t)$ and $K_2(t)$ are, respectively, the thermal conductivities for heat transfer between the working fluid and the hot source and cold reservoir. We shall assume that at time $t = 0$ the working fluid begins to connect with the hot source. Therefore, we may write $K_1(t)$ and $K_2(t)$ as

$$K_1(t) = \begin{cases} K_1 & 0 \leqslant t < t_1 \\ 0 & t_1 \leqslant t < \tau \end{cases} \tag{3}$$

$$K_2(t) = \begin{cases} 0 & 0 \leqslant t < t_1 \\ K_2 & t_1 \leqslant t < \tau \end{cases} \tag{4}$$

where t_1 is the time of heat input, K_1 and K_2 are constants.

3. Optimal cycle configuration

From the first law of thermodynamics, the total work produced in one cycle may be written as

$$W = \int_0^\tau \{K_1(t)[T_h(t) - T(t)] - K_2(t)[T(t) - T_l(t)]\}\,\mathrm{d}t \tag{5}$$

and from the second law of thermodynamics, the entropy change of the working fluid per cycle is

$$\triangle S = \int_0^\tau \{K_1(t)[T_h(t)/T(t) - 1] - K_2(t)[1 - T_l(t)/T(t)]\}\,\mathrm{d}t = 0 \tag{6}$$

because the cycle is endoreversible. Furthermore, since the heat capacities of the hot and cold reservoirs are assumed to be constants, we have

$$\dot{Q}_1 = -C_1\dot{T}_h(t) \tag{7}$$

$$\dot{Q}_2 = C_2\dot{T}_l(t). \tag{8}$$

Combining equations (1) and (7), and (2) and (8), we obtain the constraint equations on the rate of change of the temperatures with time of the hot and cold reservoirs

$$C_1\dot{T}_h(t) + K_1(t)[T_h(t) - T(t)] = 0 \tag{9}$$

$$C_2\dot{T}_l(t) + K_2(t)[T_l(t) - T(t)] = 0. \tag{10}$$

We now determine the optimal configuration of the model cycles in which the maximum work or the maximum efficiency can be obtained under a given cyclic time τ and heat input Q_1. This is equivalent to determining the minimum Q_2 under the given τ and Q_1. It is an optimization problem which may be handled by the method of Lagrange multipliers [13, 14]. For this reason, using equation (2) and the constraint equations (1), (6), (9) and (10), we introduce the modified Lagrangian

$$\begin{aligned} L = {} & K_2(t)[T(t) - T_l(t)] + \lambda_1 K_1(t)[T_h(t) - T(t)] \\ & +\lambda_2\{K_1(t)[T_h(t)/T(t) - 1] - K_2(t)[1 - T_l(t)/T(t)]\} \\ & +\mu_1(t)\{C_1\dot{T}_h(t) + K_1(t)[T_h(t) - T(t)]\} \\ & +\mu_2(t)\{C_2\dot{T}_l(t) + K_2(t)[T_l(t) - T(t)]\}. \end{aligned} \tag{11}$$

Then, from the Euler–Lagrange equations $\frac{\partial L}{\partial T(t)} = 0$, $\frac{\partial L}{\partial T_h(t)} - \frac{\mathrm{d}}{\mathrm{d}t}[\frac{\partial L}{\partial \dot{T}_h(t)}] = 0$ and $\frac{\partial L}{\partial T_l(t)} - \frac{\mathrm{d}}{\mathrm{d}t}[\frac{\partial L}{\partial \dot{T}_l(t)}] = 0$, one can obtain [13]

$$T_h(t) = \begin{cases} T_H \exp[-(K_1/C_1)(1-u)t] & 0 \leqslant t < t_1 \\ T_H \exp[-(K_1/C_1)(1-u)t_1] & t_1 \leqslant t < \tau \end{cases} \tag{12}$$

$$T_l(t) = \begin{cases} T_L & 0 \leqslant t < t_1 \\ T_L \exp[-(K_2/C_2)(1-v)(t-t_1)] & t_1 \leqslant t < \tau \end{cases} \tag{13}$$

$$T(t) = \begin{cases} uT_h(t) = uT_H \exp[-(K_1/C_1)(1-\mu)t] & 0 \leqslant t < t_1 \\ vT_l(t) = vT_L \exp[-(K_2/C_2)(1-v)(t-t_1)] & t_1 \leqslant t < \tau \end{cases} \tag{14}$$

where u and v are two constants which can be determined by the constraint equations

$$Q_1 = C_1 T_H \{1 - \exp[-(K_1/C_1)(1-u)t_1]\} \tag{15}$$

and

$$K_1(u^{-1} - 1)t_1 + K_2(v^{-1} - 1)t_2) = 0 \tag{16}$$

where $t_2 = \tau - t_1$. The results are

$$u = 1 + \frac{C_1}{K_1 t_1} \ln\left(1 - \frac{Q_1}{C_1 T_H}\right) \tag{17}$$

$$v = \frac{1 + \frac{C_1}{K_1 t_1} \ln(1 - \frac{Q_1}{C_1 T_H})}{1 + (\frac{1}{K_1 t_1} + \frac{1}{K_2 t_2}) C_1 \ln(1 - \frac{Q_1}{C_1 T_H})}. \tag{18}$$

Using equations (10), (13), (14) and (18), one can obtain

$$Q_2 = C_2 T_L \left[\exp \frac{-\frac{C_1}{C_2} \ln(1 - \frac{Q_1}{C_1 T_H})}{1 + (\frac{1}{K_1 t_1} + \frac{1}{K_2 t_2}) C_1 \ln(1 - \frac{Q_1}{C_1 T_H})} - 1\right] \tag{19}$$

and the efficiency of the cycle may be expressed as

$$\eta = 1 - \frac{Q_2}{Q_1} = 1 - \frac{C_2 T_L}{Q_1}\left[\exp \frac{-\frac{C_1}{C_2} \ln(1 - \frac{Q_1}{C_1 T_H})}{1 + (\frac{1}{K_1 t_1} + \frac{1}{K_2 t_2}) C_1 \ln(1 - \frac{Q_1}{C_1 T_H})} - 1\right]. \tag{20}$$

Equation (14) shows that the optimal cycle consists of two adiabatic processes which occur in negligible time [13, 17] and a heat absorbed and a heat released process in which the temperatures of the working fluid $T(t)$, respectively, decreases and increases exponentially with time t increasing.

4. The fundamental optimal relation

Under the circumstances of given Q_1 and the cyclic time τ, it can be shown that when

$$t_2/t_1 = \sqrt{K_1/K_2} \equiv \delta \tag{21}$$

the efficiency of the cycle attains its maximum and can be written as

$$\eta = 1 - \frac{C_2 T_L}{Q_1}\left\{\exp\left[\frac{-\frac{C_1}{C_2} \ln(1 - \frac{Q_1}{C_1 T_H})}{1 + \frac{C_1}{K\tau} \ln(1 - \frac{Q_1}{C_1 T_H})}\right] - 1\right\} \tag{22}$$

where $K = K_1 K_2/(\sqrt{K_1} + \sqrt{K_2})^2$. Then, using the relation between the power output P and the rate of heat input Q_1/τ

$$P = \eta Q_1/\tau \tag{23}$$

we can find that the optimal relation between the power output and the efficiency, i.e. the fundamental optimal relation of the cycle is given by

$$P = -K\eta Q_1 \left\{\frac{1}{C_1 \ln(1 - \frac{Q_1}{C_1 T_H})} + \frac{1}{C_2 \ln[1 + (1-\eta)\frac{Q_1}{C_2 T_L}]}\right\}. \tag{24}$$

Equation (24) is an important relation for a two finite heat source cycle, the various optimal performances of the cycle may be discussed by using it. In fact, that is just the reason why equation (24) can be called the fundamental optimal relation. Thus, we should firstly find it in the researches of finite-time thermodynamic cycle theory.

It is worthwhile to point out that the fundamental optimal relation is related to the given Q_1. It is independent of Q_1 only if $C_1 \to \infty$ and $C_2 \to \infty$. In this case, equation (24) can be written as

$$P = K\eta \left(T_H - \frac{T_L}{1-\eta} \right). \tag{25}$$

Equation (25) is just the fundamental optimal relation of an endoreversible Carnot cycle [7, 9, 16]. It is shown that all of the results derived from an endoreversible two infinite heat source cycle can be derived from equation (24), so the conclusions obtained in this paper are more general and useful than those derived from two infinite heat source cycles.

5. An example of the application of equation (24)

As an example, we consider a gas Brayton cycle under the influence of thermal resistance. For a gas Brayton cycle, two heat exchanging processes are isobaric in which the heat capacities of the gas are C_p. Thus the corresponding heat capacities of the hot and cold reservoirs are $-C_\mathrm{p}$, i.e.

$$C_1 = C_2 = -C_\mathrm{p}. \tag{26}$$

Substituting equation (26) into equation (24), the fundamental optimal relation of a gas Brayton cycle influenced by thermal resistance can be written as

$$P = K\eta \frac{Q_1}{C_\mathrm{p}} \left\{ \frac{1}{\ln(1 + \frac{Q_1}{C_\mathrm{p} T_H})} + \frac{1}{\ln[1 - (1-\eta)\frac{Q_1}{C_\mathrm{p} T_H}]} \right\}. \tag{27}$$

Moreover, from equation (14), one can see that the temperatures of the working fluid in the isobaric absorbing and releasing heat processes are proportional to those of the hot and cold reservoirs, and the proportional coefficients are u and v, respectively. Therefore, the heat capacities of the working fluid in the two isobaric processes are, respectively, C_p/u and C_p/v. Again, from equations (17), (18) and (26), one can see that $u < 1$, $v > 1$, so that $C_\mathrm{p}/u > C_\mathrm{p}$, $C_\mathrm{p}/v < C_\mathrm{p}$. This shows that the form of the optimal cycle for a Brayton cycle under the influence of thermal resistance is no longer a Brayton cycle, shown as figure 2. This is characteristic of a two finite heat source cycle. It is different from a Carnot cycle. For a Carnot cycle under the influence of thermal resistance, the form of optimal cycle is still a Carnot cycle. These results provide some new theoretical basis for the optimal design of real heat engines.

6. Discussion

(1) When $K_1 \to \infty$ and $K_2 \to \infty$, i.e. the influence of thermal resistance can be neglected, the cycle is reversible, equation (24) changes into

$$\eta = 1 - \frac{C_2 T_L}{Q_1} \left[\left(1 - \frac{Q_1}{C_1 T_H} \right)^{-C_1/C_2} - 1 \right] \equiv 1 - \frac{T_L}{T_H^*} \equiv \eta_\mathrm{c}^* \tag{28}$$

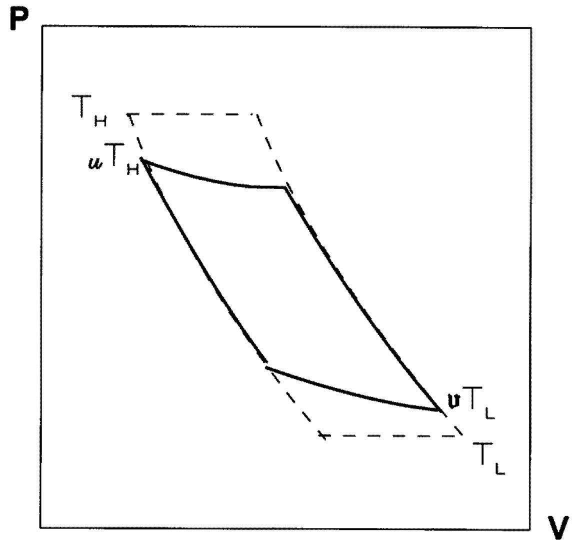

Figure 2. The pressure–volume schematic diagram of a gas Brayton cycle influenced by thermal resistance.

where η_c^* is the efficiency of a reversible two finite heat source cycle, and

$$T_H^* = \frac{Q_1}{C_2}\left[\left(1 - \frac{Q_1}{C_1 T_H}\right)^{-C_1/C_2} - 1\right]^{-1} \tag{29}$$

is the temperature of the hot reservoir of an equivalent Carnot cycle of the reversible two finite heat source cycle, and the temperature of the cold reservoir of the equivalent Carnot cycle is T_L.

(2) When $C_2 \to \infty$ and C_1 is finite, we can obtain

$$P = -K\eta\left[\frac{Q_1}{C_1 \ln(1 - \frac{Q_1}{C_1 T_H})} + \frac{T_L}{1-\eta}\right] = K\eta\left(T_H^\dagger - \frac{T_L}{1-\eta}\right) \tag{30}$$

from equation (24), where

$$T_H^\dagger \equiv \lim_{C_2\to\infty} T_H^* = -Q_1 \Big/ \left[C_1 \ln\left(1 - \frac{Q_1}{C_1 T_H}\right)\right]. \tag{31}$$

If $K \to \infty$, equation (30) can be simplified to

$$\eta = 1 - T_L/T_H^\dagger. \tag{32}$$

If $K \to \infty$ and $C_1 \to \infty$, $T_H^\dagger = T_H$ and $\eta = \eta_c$ (Carnot efficiency). Equations (30) and (32) show that $T_H^\dagger$ concentrically reflects the effect of finite heat capacity C_1 on performance of a cycle with a finite heat source and an infinite heat sink.

Equation (30) is just the fundamental optimal relation of an endoreversisle cycle with a finite heat source and an infinite heat sink had been derived in [14], and equation (32) is just the efficiency expression of a reversible cycle with a finite heat source and an infinite heat sink [18]. This shows again that equation (24) is a quite general and important relation. Many of the main results in relevant references may be deduced from it.

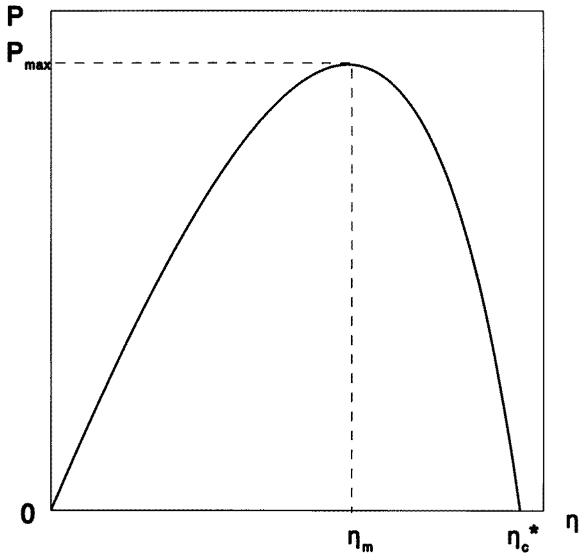

Figure 3. The power output against efficiency curve of an endoreversible two finite heat source cycle.

(3) When $C_1 \to \infty$ and C_2 is finite, one can obtain

$$P = K\eta \left\{ T_H - \frac{Q_1}{C_2 \ln[1 + (1-\eta)\frac{Q_1}{C_2 T_L}]} \right\} = K\eta \left(T_H - \frac{T_L^{\dagger}}{1-\eta} \right) \tag{33}$$

from equation (24), where

$$T_L^{\dagger} = \frac{Q_2}{C_2 \ln(1 + \frac{Q_2}{C_2 T_L})} \tag{34}$$

is the temperature of the cold heat reservoir of an equivalent Carnot cycle for a given Q_2, which concentrically reflects the effect of finite heat capacity C_2 on performance of a cycle with an infinite heat source and a finite heat sink.

If C_2 is also infinite, $T_L^{\dagger} = T_L$. This shows again that equation (24) is also suitable for reversible Carnot cycles.

(4) The characteristic curve of the optimal relation between the power output P and efficiency η of a two finite heat source cycle described by equation (24) is shown in figure 3. From figure 3, one can see that the characteristic curve has two zero power points and a maximum power point. The efficiencies of the two zero power points are, respectively, $\eta_1 = 0$ and $\eta_2 = \eta_c^*$. The maximum power $P_{\max}$ and its corresponding efficiency η_m can be obtained from equation (24) and the extremal condition $\partial P/\partial\eta = 0$. According to the extremal condition and equation (24), we have the following equation

$$\begin{aligned} &\left[1 + (1-\eta)\frac{Q_1}{C_2 T_L}\right] \left\{ \ln\left[1 + (1-\eta)\frac{Q_1}{C_2 T_L}\right] \right\}^2 + \frac{C_1}{C_2}\left[1 + (1-\eta)\frac{Q_1}{c_2 T_L}\right] \\ &\quad \times \ln\left(1 - \frac{Q_1}{C_1 T_H}\right) \ln\left[1 + (1-\eta)\frac{Q_1}{C_2 T_L}\right] + \eta \frac{Q_1}{C_2 T_L}\frac{C_1}{C_2} \ln\left(1 - \frac{Q_1}{C_1 T_H}\right) = 0. \end{aligned} \tag{35}$$

Equation (35) is a transcendental equation. There is no analytic solution in general, but it can be solved by using numerical or graphic method when one needs.

Similar to endoreversible Carnot cycle, η_m determines the lower bound of the efficiency for a two finite heat source cycle. When $\eta < \eta_m$, the efficiency η and power output P of the cycle are less than those at the maximum power point. Therefore, such states are not the optimal operating state. Thus, a new criterion can be established for the selection of an optimal operating parameter for a two finite heat source cycle, that is, the efficiency η of the cycle should satisfy the following equation:

$$\eta_m \leqslant \eta < \eta_c^*. \tag{36}$$

It is worthwhile to point out that the η_m here is different from that of an endoreversible Carnot cycle. In general, it is not equal to $1 - \sqrt{T_L/T_H}$ or $1 - \sqrt{T_L/T_H^*}$, only when $C_2 \to \infty$, $\eta_m = 1 - \sqrt{T_L/T_H^*}$, and only when $C_1 \to \infty$ and $C_2 \to \infty$, $\eta_m = 1 - \sqrt{T_L/T_H}$.

(5) From equation (24), the various optimal relations and the bounds of performances of the two finite heat source endoreversible cycle can be derived, so that the optimal performances of the cycle can be discussed. For example, from equation (24), we can derive the optimal relation between the rate of entropy production σ and the efficiency η of the cycle as follows:

$$\sigma = K \left\{ \frac{1}{C_1 \ln(1 - \frac{Q_1}{C_1 T_H})} + \frac{1}{C_2 \ln[1 + (1-\eta)\frac{Q_1}{C_2 T_L}]} \right\} \times \left\{ C_1 \ln\left(1 - \frac{Q_1}{C_1 T_H}\right) + C_2 \ln\left[1 + (1-\eta)\frac{Q_1}{C_2 T_L}\right] \right\}. \tag{37}$$

Again, from equations (37) and (24), we can obtain the minimum rate of entropy production for the cycle at a given power output. This is the minimum irreversible loss which cannot be avoided for an endoreversible two finite heat source cycle at a given power output. The main difference between finite-time thermodynamics and classical termodynamics just lies in that the minimum unavoidable irreversible loss in a thermodynamic process can be found in the former but not in the later. Therefore, finite-time thermordynamics is more significant and useful for real thermodynamic processes, and the conclusions obtained in this paper may contribute to a better understanding of optimal performance of real two finite heat source cycles.

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